June 1964

INVARIANT FUNCTIONS ON SAMPLE AND PARAMETER SPACES*

R. B. Hora

Technical Report No. 45

*This research was supported in part by the National Science Foundation under Grant Number G-19126.

1. Introduction. Lehmann (1959) has discussed invariant functions on sample space and obtained theorems concerning: (1) relationship of maximal invariant with an invariant function, (2) a method for obtaining maximal invariant and (3) as to how the parameter space can be shrunk by use of maximal invariant on parameter space.

In this paper we consider invariant functions on sample and parameter spaces and define the concept of maximal invariant in a manner which is an obvious extension of Lehmann's definition. It may be pointed out that whereas the invariant functions defined by Lehmann (1959) are useful for testing of hypotheses problems, the invariant functions defined in this paper (as noted in Section 5) are useful for the purposes of the problems of invariant estimation and prediction.

We give (1) a relationship corresponding to that of Lehmann between a maximal invariant and an invariant function and (2) a general expression for maximal invariant. We conclude the paper by giving some examples of maximal invariant functions for certain invariant specifications and an example of a function which is invariant but not maximal invariant.

2. Invariant and maximal invariant functions. To define invariant and maximal invariant functions on sample spaces it is necessary to make the following assumptions.

Assumption 1. Let $(\mathfrak{X},B_X)$ be a measurable space, where $\mathfrak{X}$ is the sample space, and $\mathcal{G} = \{g\}$ be a group of one-to-one measurable transformations of $\mathfrak{X}$ onto itself.

Assumption 2. For each ω in the parameter space Ω, P^{ω} is a probability measure on $(\mathfrak{X},B_X)$ such that for each $g\in\mathcal{G}$ and each $\omega\in\Omega$ there exists a unique $\omega_g\in\Omega$ for which

(2.1) $$P^{\omega}(X) = P^{\omega_g}(gX) \quad \text{all } X\in B_X$$

Let g^* be the one-to-one function of Ω onto Ω defined by

(2.2) $$g^*\omega = \omega_g .$$

It is easily seen that $\mathcal{G}^* = \{g^*\}$ if a group of transformations of Ω which is isomorphic to $\mathcal{G}$ when $g_1^* \cdot g_2^*$ is defined by $(g_1^* \cdot g_2^*)\omega = \omega_{g_1 \cdot g_2}$.

Assumption 3. $\mathcal{G}^*$ is exactly transitive on Ω, that is, for any $\omega_1, \omega_2 \in \Omega$ there is a unique $g^* \in \mathcal{G}^*$ such that $g^*\omega_1 = \omega_2$.

It may be remarked that these assumptions correspond to the first three assumptions made in Hora and Buehler (1964).

We now proceed to define invariant and maximal invariant functions on sample and parameter spaces for which the above assumptions hold.

Definition 2.1. Let H be a function on $\mathcal{X} \times \Omega$ such that for all $g, g^* (g \longleftrightarrow g^*)$

(2.3) $\quad H(x,\omega) = H(gx, g^*\omega)$. Then H will be called an invariant function on $\mathcal{X} \times \Omega$ under $\mathcal{G}$.

Definition 2.2. Let S be an invariant function on $\mathcal{X} \times \Omega$ under $\mathcal{G}$ such that if

(2.4) $\quad S(x,\omega_1) = S(x',\omega_2)$ then there exists a $g \in \mathcal{G}$ and $g^* \in \mathcal{G}^* (g \longleftrightarrow g^*)$ such that $gx = x'$ and $g^*\omega_1 = \omega_2$. Then $S(x,\omega)$ will be called a maximal invariant under $\mathcal{G}$.

3. Some theorems concerning invariant functions. In this Section we give two theorems concerning invariant functions; the first gives a characterization of invariant functions and the second gives a general expression for a maximal invariant for specifications for which the assumptions of Section 2 hold. The proof of the first is based on an obvious generalization of the argument used by Lehmann (1959) in proving the corresponding theorem for invariant functions on sample space. However, Lehmann (1959) gives no general method for obtaining maximal invariant on sample space. Instead he points out that frequently, it is convenient to obtain a maximal invariant in steps, each corresponding to a sub-group of the group of transformations of the sample space. Indeed it is not possible to obtain an analog of our second theorem for maximal invariant on sample space.

Theorem 3.1. Assume that assumptions 1-3 of Section 2 hold and $S(x,\omega)$ is a maximal invariant. Then a necessary and sufficient condition for $H(x,\omega)$ to be invariant is that it depends on x,ω only through $S(x,\omega)$, i.e. there exists a function h such that $H(x,\omega) = h[S(x,\omega)]$, for all x,ω.

Proof: Let $H(x,\omega) = h[S(x,\omega)]$, for all x,ω. Then $H(gx,g^*\omega) = h[S(gx,g^*\omega)] = H(x,\omega)$ for $S(x,\omega)$ is a maximal invariant. Conversely, if $H(x,\omega)$ is invariant and $S(x,\omega_1) = S(x',\omega_2)$, then for some $g \in \mathcal{G}$ and $g^* \in \mathcal{G}^*$, $(g \longleftrightarrow g^*)$, $gx = x'$ and $g^*\omega_1 = \omega_2$. Therefore $H(x,\omega_1) = H(x',\omega_2)$.

Theorem 3.2. Assume that assumptions 1-3 of Section 2 hold and let $S(x,\omega) = g_\omega^{-1}x$, where g_ω is the element of $\mathcal{G}$ that corresponds to $\omega \epsilon \Omega$. (Refer to (2.8) on page 4 of Hora and Buehler (1964)). Then $S(x,\omega)$ is a maximal invariant.

Proof: $S(g_{\omega_1}x, g^*_{\omega_1}\omega) = (g_{\omega_1} \cdot g_\omega)^{-1}\, g_{\omega_1} x = g_\omega^{-1}x$. Hence $S(x,\omega)$ is invariant. Also, let $S(x,\omega_1) = S(x',\omega_2)$. Take $g^* = g^*_{\omega_2} \cdot g^{*-1}_{\omega_1}$ and thus the corresponding $g = g^{-1}_{\omega_2} \cdot g_{\omega_1}$. Then $g^*_{\omega_2} \cdot g^{*-1}_{\omega_1}\omega_1 = \omega_2$ and $g_{\omega_2} \cdot g^{-1}_{\omega_1} x = x'$ for $g^{-1}_{\omega_1} x = g^{-1}_{\omega_2} x'$, by hypothesis. Hence $S(x,\omega)$ is maximal invariant.

4. Examples. In this Section, by use of Theorem 3.2, we obtain maximal invariants for examples of invariant specifications considered in Section 3 of Hora and Buehler (1964). Besides we also give the maximal invariant for an invariant specification concerning bivariate distribution considered by Fraser (1963). We conclude this Section by giving an example of a function which is invariant but not maximal invariant.

We give below in Table 4.1, expressions for maximal invariants. These expressions will be given only in terms of x_1 and y_1 and it is to be understood that expressions are similar in other x's and y's.

TABLE 4.1

Case	Maximal Invariant $S = g_{\omega}^{-1} x$
θ	$x_1 - \theta$
θ,σ	$\frac{x_1-\theta}{\sigma}$
θ_1,θ_2,σ	$\frac{x_1-\theta_1}{\sigma}, \frac{y_1-\theta_2}{\sigma}$
$\theta_1,\sigma_1,\theta_2,\sigma_2$	$\frac{x_1-\theta_1}{\sigma_1}, \frac{y_1-\theta_2}{\sigma_2}$
# $\theta_1,\theta_2,\sigma_1,\sigma_2,\rho$	$-\frac{\theta_1}{\sigma_1} + \frac{1}{\sigma_1}x_1, (\frac{\sigma_2}{\sigma_1}\rho - \theta_2)\frac{1}{\sigma_2\sqrt{1-\rho^2}} - \frac{\rho}{\sigma_1\sqrt{1-\rho^2}} x_1 + \frac{1}{\sigma_2\sqrt{1-\rho^2}} y_1$

#For details of this case concerning the definition of g etc., the reader is referred to Fraser (1963).

The appropriate g (and hence g^*) of Definition 2.2 can be easily obtained. For example in θ,σ case $g = g_{\alpha,\beta}$ where

$$\alpha = \frac{x_1x_2' - x_2x_1'}{x_1 - x_2} \quad \text{and} \quad \beta = \frac{x_2' - x_1'}{x_2 - x_1}\ ; \quad \text{and}$$

in θ_1,θ_2,σ case $g = g_{\alpha_1,\alpha_2,\beta}$ where

$$\alpha_1 = \frac{x_1x_2' - x_2x_1'}{x_1 - x_2}\ , \alpha_2 = y_1' - \frac{x_2' - x_1'}{x_2 - x_1} y_1 \quad \text{and} \quad \beta = \frac{x_2' - x_1'}{x_2 - x_1}\ .$$

Actually one obtains a g by equating an appropriate number of components of x and x' and then verifying that this g is appropriate i.e. gx = x' holds. This verification except in simple situations like the θ case involves combersome algebra. It may also be remarked that α's and β's used to define g do not necessarily have to be defined in terms of the first components of x and x'.

Example of a function which is invariant but not maximal invariant. Consider the case of one location parameter family of distributions. Let $F(x,\theta) = (x_1-\theta)^2 + (x_2-\theta)^2$. Take $x = (4,5)$ and $x' = (0,5)$, $\theta_1 = 1$ and $\theta_2 = 0$. Then $F(x,\theta_1) = F(x',\theta_2) = 25$. However, there is no g and g^*, $(g \longleftrightarrow g^*)$ such that $gx = x'$ and $g^*\theta_1 = \theta_2$. Thus $F(x,\theta)$ is not maximal invariant but it can be easily verified that it is an invariant function.

5. Applications of invariant functions.

Invariant functions play a vital role in the theory of invariant estimation and invariant prediction. Possibly, such functions were first used by Pitman (1939) for invariant estimation and by Ramsey and Buehler (1963) for invariant prediction. However, they did not give any formal consideration to such functions nor did they give any group-theoretic definition as is given by us in this paper. Reference to such functions may also be found in Hora (1964) and Hora and Buehler (1964).

In the second of these references it is seen that in the notation of that paper, functions $\varphi(g^{*-1}\hat{\psi}(x))$ which are useful for the purposes of estimation are invariant. Furthermore, $H(x,\omega) = g^{+-1}\hat{\psi}(x) = \hat{\psi}(g_\omega^{-1}x)$. Thus $H(x,\omega)$ is a function of the maximal invariant.

References

Fraser, D.A.S. (1963). On the definition of fiducial probability.

Hora, R. B. (1964). Fiducial probability theory for distributions with a group structure. Technical Report No. 33, Dept. of Stat., Univ. of Minn., Minneapolis

Hora, R. B. and Buehler, R. J. (1964). Fiducial theory and invariant estimation. Technical Report No. 42, Dept. of Stat., Univ. of Minn., Minneapolis.

Lehmann, E. L. (1959). Testing Statistical Hypotheses. Wiley, New York.

Pitman, E. J. G. (1939). The estimation of the location and scale parameters of a continuous population of any given form, Biometrika 30 pp. 391-421.

Ramsey, F. L. and Buehler, R. J. (1963). Prediction in location and scale parameters families. Tech. Rep. No. 2, NSF, Stat. Lab., Iowa State Univ., Ames.